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Automorphisms of the Lie algebra of strictly upper triangular matrices over certain commutative rings ☆

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Abstract

Let $\mathbf{n}$ be the nilpotent Lie algebra consisting of all strictly upper triangular $(n+1) \times (n+1)$ matrices over a commutative ring R. In this paper, we discuss the automorphism group of $\mathbf{n}$. We prove that any automorphism φ of $\mathbf{n}$ can be uniquely expressed as $\varphi = \omega \cdot \eta \cdot \xi \cdot \mu \cdot \sigma$, where ω, η, ξ, μ and σ are graph, diagonal, external, central and inner automorphisms, respectively, of $\mathbf{n}$ when $n \geqslant 3$ and R is a local ring that contains 2 as a unit or an integral domain of characteristic other than two. In the case $n = 2$ we also prove that any automorphism of $\mathbf{n}$ can be expressed as a product of graph, diagonal, extremal and inner automorphisms for an arbitrary local ring R. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Significant research has been done in studying automorphisms of matrix algebras and their subalgebras, see [1,5–8]. Let $M_n(R)$ be the R-algebra of all $n \times n$ matrices over a commutative ring R. The bracket operation $[x, y] = xy - yx$ defines on $M_n(R)$ a structure of Lie algebra over R. Let $\mathbf{t}$ and $\mathbf{b}$ be the solvable Lie subalgebras

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of this Lie algebra consisting of all upper triangular matrices and all upper triangular matrices of trace 0, respectively. The automorphism groups of $\mathbf{t}$ and $\mathbf{b}$ have been determined by Doković [3] and Cao [2].

Let R be a commutative ring with identity and $\mathbf{n}$ the nilpotent Lie subalgebra of the Lie algebra $M_{n+1}(R)$ consisting of all strictly upper triangular matrices. In this paper, we will discuss the automorphism group of $\mathbf{n}$. We prove that any automorphism φ of $\mathbf{n}$ can be uniquely expressed as $\varphi = \omega \cdot \eta \cdot \xi \cdot \mu \cdot \sigma$, where $\omega,\ \eta,\ \xi,\ \mu$ and σ are graph, diagonal, extremal, central and inner automorphisms, respectively, of $\mathbf{n}$ when $n \geqslant 3$ and R is a local ring that contains 2 as a unit or an integral domain of characteristic other than two. In the case $n = 2$ we also prove that any automorphism of $\mathbf{n}$ can be expressed as a product of graph, diagonal, extremal and inner automorphisms for an arbitrary local ring R.

In Section 2, we give some necessary notations and preliminary results. In Section 3, we define five types of automorphisms of $\mathbf{n}$ called *standard automorphisms*, which build the automorphism group of $\mathbf{n}$ under our conditions for R. The descriptions of the automorphism group of $\mathbf{n}$, main results of this paper and their proofs are given for $n \geqslant 3$ in Section 4 and for $n = 1, 2$ in Section 5, respectively.

For the definitions of the standard automorphisms and the main results of this paper, some ideas arise from Gibbs [4], where the automorphisms of certain unipotent subgroups of Chevalley groups and Steinberg groups over a field are discussed.

2. Preliminaries

Let R be a commutative ring with identity and R^* the group of invertible elements of R. Let $M_{n+1}(R)$ be the Lie algebra of $(n+1) \times (n+1)$ matrices over R, where n is a positive integer. Denote by $\mathbf{n}$ the nilpotent Lie subalgebra of $M_{n+1}(R)$ consisting of all strictly upper triangular matrices. Let e be the identity matrix in $M_{n+1}(R)$ and e_{ij} the matrix in $M_{n+1}(R)$ whose sole nonzero entry is 1 in the (i, j) position. It is well known that the matrix set $\{e_{ij} \mid 1 \leqslant i < j \leqslant n+1\}$ is a basis of $\mathbf{n}$ and for any x in $\mathbf{n}$, we can write $x = \sum_{i<j} a_{ij} e_{ij}$ for $a_{ij} \in R$. For convenience sake, in this expression the subscript i can be less than 1 and j can be greater than $n+1$ and we use the convention that the coefficient a_{ij} is regarded as zero if $i < 1$ or $j > n+1$ in some term $a_{ij} e_{ij}$.

Let

$$\mathbf{n}_1 = \mathbf{n}, \quad \mathbf{n}_2 = [\mathbf{n}, \mathbf{n}_1], \quad \mathbf{n}_3 = [\mathbf{n}, \mathbf{n}_2], \quad \ldots$$

be the lower central series of $\mathbf{n}$. Each $\mathbf{n}_k$ is an ideal of $\mathbf{n}$ and is invariant under any automorphism of $\mathbf{n}$. It is easy to see that $\mathbf{n}_k = \sum_{j-i \geqslant k} Re_{ij}$ and the center of the Lie algebra $\mathbf{n}$ is $\mathbf{n}_n = Re_{1,n+1}$. It is easy to check that

$$\mathbf{n}_k \mathbf{n}_l = \{xy \mid x \in \mathbf{n}_k,\ y \in \mathbf{n}_l\} \subseteq \mathbf{n}_{k+l}$$

and

$$[\mathbf{n}_k, \mathbf{n}_l] \subseteq \mathbf{n}_{k+l}.$$

We denote by Aut($\mathbf{n}$) the automorphism group of the Lie algebra $\mathbf{n}$ and by 1 both the identity subgroup of Aut($\mathbf{n}$) and the identity automorphism of $\mathbf{n}$.

If $\mathscr{A}, \mathscr{B}$ are two subgroups of a group, we use $\mathscr{A}\mathscr{B}$, $\mathscr{A} \ltimes \mathscr{B}$ and $\mathscr{A} \times \mathscr{B}$ to denote their product, semidirect product with $\mathscr{B}$ normal and direct product, respectively.

Lemma 2.1. *Let φ be in* Aut($\mathbf{n}$). *Then*
(i) $\varphi(\mathbf{n}_k \backslash \mathbf{n}_{k+1}) = \mathbf{n}_k \backslash \mathbf{n}_{k+1}, \quad k = 1, 2, \ldots,$
(ii) $\varphi(e_{1,n+1}) = ae_{1,n+1}$ *with some* $a \in R^*$.

Proof. (i) Since $\varphi(\mathbf{n}_k) \subseteq \mathbf{n}_k$, and $\varphi^{-1}(\mathbf{n}_k) \subseteq \mathbf{n}_k$, we have $\varphi(\mathbf{n}_k) = \mathbf{n}_k$. It is clear that $\varphi(\mathbf{n}_k \backslash \mathbf{n}_{k+1}) = \mathbf{n}_k \backslash \mathbf{n}_{k+1}$. (ii) It is clear that φ induces an automorphism of the free R-module $\mathbf{n}_n$ of rank 1. So the assertion is true. $\square$

Lemma 2.2. *Let φ be in* Aut($\mathbf{n}$) *and*

$$\varphi(e_{i,i+1}) \equiv \sum_{j=1}^{n} a_{ji} e_{j,j+1} \bmod \mathbf{n}_2, \quad i = 1, \ldots, n.$$

Set

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

Then $\det A \in R^*$.

Proof. The assertion follows from the fact that the R-module $\mathbf{n}_1/\mathbf{n}_2$ is a free of rank n and φ induces an automorphism of that module. $\square$

Lemma 2.3. *Assume that R be any commutative ring. Moreover assume that* 2 *is not a zero divisor of R if $n \geqslant 3$. Then for any $\varphi \in$* Aut($\mathbf{n}$), *any $y \in \mathbf{n}$ and any basis element e_{ij} of $\mathbf{n}$ we have $\varphi(e_{ij})^2 y = 0$, $y\varphi(e_{ij})^2 = 0$ and $\varphi(e_{ij})y\varphi(e_{ij}) = 0$.*

Proof. Applying φ to $[e_{ij}, [e_{ij}, \varphi^{-1}(e_{kl})]] = 0$, where e_{kl} is any basis element of $\mathbf{n}$, we obtain $\varphi(e_{ij})^2 e_{kl} + e_{kl}\varphi(e_{ij})^2 - 2\varphi(e_{ij})e_{kl}\varphi(e_{ij}) = 0$. Since $\varphi(e_{ij})^2 e_{kl} \in \sum_{r<k} Re_{rl}$, $e_{kl}\varphi(e_{ij})^2 \in \sum_{s>l} Re_{ks}$ and $\varphi(e_{ij})e_{kl}\varphi(e_{ij}) \in \sum_{r<k,s>l} Re_{rs}$, we obtain $\varphi(e_{ij})^2 e_{kl} = 0$, $e_{kl}\varphi(e_{ij})^2 = 0$ and $2\varphi(e_{ij})e_{kl}\varphi(e_{ij}) = 0$. When $n = 1, 2$, it is trivial that $\varphi(e_{ij})e_{kl}\varphi(e_{ij}) = 0$. When $n \geqslant 3$, this is also true since 2 is not a zero divisor. It follows that $\varphi(e_{ij})^2 y = 0$, $y\varphi(e_{ij})^2 = 0$ and $\varphi(e_{ij})y\varphi(e_{ij}) = 0$ for any $y \in \mathbf{n}$. $\square$

3. The standard automorphisms of n

In this section, we will give five types of standard automorphisms of **n**, which build the automorphism group Aut(**n**) under our conditions for R. The standard automorphisms of **n** are as follows.

3.1. Inner automorphisms

For any z in **n**, $x = e + z$ is invertible and the map $\sigma_x : y \mapsto xyx^{-1}$ is an automorphism of **n**, which is called an *inner automorphism*. The set of all inner automorphisms of **n** is a subgroup of Aut(**n**), which is called the *inner automorphism group* of **n** and denoted by $\mathscr{I}$. It is known that the matrix $x = e + z$ with $z \in \mathbf{n}$ can be expressed as a product of some matrices of the form $e + ae_{ij}$ for $a \in R$ and $1 \leqslant i < j \leqslant n + 1$. So the inner automorphism group $\mathscr{I}$ is generated by the inner automorphisms of the form σ_x with $x = e + ae_{ij}$. (Furthermore, the set of matrices $e + z$ with $z = ae_{i,i+1},\ a \in R,\ i = 1, \ldots, n$, is also a set of generators of the group $\mathscr{I}$. It follows from the formula $e + ae_{ij} = (e + ae_{ik})(e + e_{kj})(e - ae_{ik})(e - e_{kj})$ with $i < k < j$.)

Lemma 3.1. *Let R be as in Lemma* 2.3. *Then* $\mathscr{I} \lhd \mathrm{Aut}(\mathbf{n})$.

Proof. It suffices to prove that $\varphi\sigma_x\varphi^{-1} \in \mathscr{I}$ for any $\varphi \in \mathrm{Aut}(\mathbf{n})$ and any $x = e + ae_{ij}$. First, we assert that if $z = e + a\varphi(e_{ij})$, then $z^{-1} = e - a\varphi(e_{ij}) + a^2\varphi(e_{ij})^2$. In fact, by Lemma 2.3 it is trivial to check that $(e + a\varphi(e_{ij}))(e - a\varphi(e_{ij}) + a^2\varphi(e_{ij})^2) = e$. Next, we prove that $\varphi\sigma_x\varphi^{-1} = \sigma_z$ with $z = e + a\varphi(e_{ij})$. For any $y \in \mathbf{n}$, by Lemma 2.3 we have

$$\begin{aligned}\varphi\sigma_x\varphi^{-1}(y) &= \varphi(\varphi^{-1}(y) + a[e_{ij}, \varphi^{-1}(y)]) \\ &= y + a[\varphi(e_{ij}), y] \\ &= (e + a\varphi(e_{ij}))y(e - a\varphi(e_{ij}) + a^2\varphi(e_{ij})^2) \\ &= \sigma_z(y)\end{aligned}$$

Hence, $\varphi\sigma_x\varphi^{-1} \in \mathscr{I}$. □

Let

$$T = \left\{ e + \sum_{i<j} a_{ij}e_{ij} | a_{ij} \in R \right\}$$

and

$$T_n = \{e + a_{1,n+1}e_{1,n+1} | a_{1,n+1} \in R\}$$

be subgroups of the general linear group $\mathrm{GL}_{n+1}(R)$. We have the following lemma.

Lemma 3.2. $\mathscr{I} \cong T/T_n$.

Proof. It is clear that the map $\rho : T \to \mathscr{I}, x \mapsto \sigma_x$, is a group epimorphism and $T_n \subseteq \mathrm{Ker}\ \rho$. Conversely, let $x = e + \sum_{i<j} a_{ij} e_{ij} \in \mathrm{Ker}\rho$. Then for $1 \leqslant k < l \leqslant n+1$ we have $x e_{kl} = e_{kl} x$, which implies that

$$\sum_{i<k} a_{ik} e_{il} = \sum_{j>l} a_{lj} e_{kj}.$$

Then it follows that $a_{ij} = 0$ if $j - i < n$, and so $x = e + a_{1,n+1} e_{1,n+1} \in T_n$. Hence $\mathrm{Ker}\rho = T_n$. The proof is completed. $\square$

It is clear that the linear automorphism group of $\mathbf{n}$ is trivial when $n = 1$.

3.2. *Diagonal automorphisms*

We denote by D the subgroup of $\mathrm{GL}_{n+1}(R)$ consisting of all diagonal matrices. For any $d \in D$, the map $\eta_d : x \mapsto dxd^{-1}$ is an automorphism of $\mathbf{n}$, which is called a *diagonal automorphism*. It is clear that $\eta_d \eta_{d'} = \eta_{dd'}$ for d and d' in D. So the set of all diagonal automorphisms of $\mathbf{n}$ is a subgroup of Aut($\mathbf{n}$), which is called the *diagonal automorphism group* of $\mathbf{n}$ and denoted by $\mathscr{D}$.

It is easy to prove the following lemma.

Lemma 3.3. $\mathscr{D} \cong D/R^* e$.

3.3. *Central automorphisms*

Let $f : \mathbf{n} \to R$ be a linear map such that $f(y) = 0$ for any $y \in \mathbf{n}_2$. It is trivial to check that the map $x \mapsto x + f(x) e_{1,n+1}$ is an automorphism of $\mathbf{n}$ when $n > 1$. And when $n = 1$ the above map is an automorphism of $\mathbf{n}$ if and only if $1 + f(e_{12}) \in R^*$ by Lemma 2.1(ii). An automorphism of $\mathbf{n}$ of this form is called a *central automorphism*. A central automorphism is called *improper* if it is also a diagonal or an inner automorphism.

When $n = 1$ a central automorphism associated to f is just the diagonal automorphism η_d with $d = \mathrm{diag}\{1 + f(e_{12}), 1\}$ and so any central automorphism is improper.

When $n \geqslant 2$ the operation of a central automorphism on the given basis of $\mathbf{n}$ is $e_{i,i+1} \mapsto e_{i,i+1} + c_i e_{1,n+1}$ for $1 \leqslant i \leqslant n$ and $e_{ij} \mapsto e_{ij}$ otherwise. Hence, it uniquely determines an n- tuple $c = (c_1, \ldots, c_n) \in R^n$. Conversely, any $c = (c_1, \ldots, c_n) \in R^n$ determines a central automorphism of $\mathbf{n}$. We denote it by μ_c. It is clear that $\mu_c \mu_{c'} = \mu_{c+c'}$ for c and c' in R^n. For any $c = (c_1, \ldots, c_n) \in R^n$, we have

$\mu_c = \mu_{c'}\mu_{c''} = \sigma_x \mu_{c''}$, where $c' = (c_1, 0, \ldots, 0, c_n)$, $c'' = (0, c_2, \ldots, c_{n-1}, 0)$ and $x = e - c_1 e_{2,n+1} + c_n e_{1n}$. Therefore, when $n = 2$ any central automorphism is improper and when $n > 2$ we need only consider those μ_c, which is called *proper*, with $c = (0, c_2, \ldots, c_{n-1}, 0)$. For a proper central automorphism we write $c = (c_2, \ldots, c_{n-1})$ for $(0, c_2, \ldots, c_{n-1}, 0)$. When $n > 2$, the set of all proper central automorphisms of $\mathbf{n}$ is a subgroup of Aut($\mathbf{n}$), which is called the *central automorphism group* of $\mathbf{n}$ and denoted by $\mathscr{C}$. This subgroup is isomorphic to the additive group R^{n-2}.

3.4. Graph automorphisms

For $x \in M_{n+1}(R)$ let x' denote the transpose of x. Set $r = e_{1,n+1} + e_{2n} + \cdots + e_{n2} + e_{n+1,1}$. It is clear that $r^2 = e$ and $r' = r$. The map $\omega_0 : x \mapsto -rx'r$ is an automorphism of $\mathbf{n}$. Referring to a symmetry of the Dynkin diagram of the complex simple Lie algebra A_n, we call ω_0 a *graph automorphism*. When $n = 1$ ω_0 is just the diagonal automorphism η_d with $d = \mathrm{diag}\{-1, 1\}$. For convenience sake, the identity automorphism of $\mathbf{n}$ is also regarded as a graph automorphism. The graph automorphism ω_0 generates a subgroup of Aut($\mathbf{n}$) of order 2, which is called the *graph automorphism group* of $\mathbf{n}$ and is denoted by $\mathscr{G}$.

3.5. Extremal automorphisms

Assume that $n \geqslant 3$. Let $b = (b_1, b_2) \in R^2$. The linear map $\xi_b : \mathbf{n} \to \mathbf{n}$ defined by $e_{12} \mapsto e_{12} + b_1 e_{2,n+1}$, $e_{n,n+1} \mapsto e_{n.n+1} + b_2 e_{1n}$ and $e_{ij} \mapsto e_{ij}$ otherwise, determines an automorphism of $\mathbf{n}$. Referring to Gibbs [4], we call it an *extremal automorphism*. It is clear the $\xi_b \xi_{b'} = \xi_{b+b'}$ for b and b' in R^2. Hence the set of all extremal automorphisms of $\mathbf{n}$ is a subgroup of Aut($\mathbf{n}$), which is called the *extremal automorphism group* of $\mathbf{n}$ and is denoted by $\mathscr{E}$. This subgroup is isomorphic to the additive group R^2.

When $n = 2$, we also denote by ξ_b the linear map defined above. Applying it to $[e_{12}, e_{23}] = e_{13}$, we obtain $\xi_b(e_{13}) = (1 - b_1 b_2) e_{13}$. By Lemma 2.1(ii) ξ_b is an automorphism of $\mathbf{n}$ if and only if $1 - b_1 b_2 \in R^*$. We also call it an extremal automorphism. However, in this case the set of all extremal automorphisms of $\mathbf{n}$ is not a subgroup of Aut($\mathbf{n}$).

4. The automorphism group of n for $n \geqslant 3$

Theorem 4.1. *Assume that $n \geqslant 3$ and R is a local ring that contains 2 as a unit or an integral domain of characteristic other than 2. Let φ be an arbitrary automorphism of* $\mathbf{n}$. *Then there are graph, diagonal, extremal, central and inner automorphisms ω, η, ξ, μ and σ, respectively, of* $\mathbf{n}$ *such that $\varphi = \omega \cdot \eta \cdot \xi \cdot \mu \cdot \sigma$.*

It is convenient to divide the proof of this theorem into the following three lemmas, in which R and φ is assumed to be as in Theorem 4.1 and A as in Lemma 2.2.

Lemma 4.2. *The matrix A is a monomial matrix, in which the sole nonzero element in the first column is a_{11} or a_{n1}.*

Proof. We first prove the first assertion. Since $\det A \in R^*$, each column of A contains a nonzero entry. Morever, if R is a local ring, then each column of A contains a unit of R; otherwise $\det A$ is in the unique maximal ideal of R, a contradiction. Let $a_{ki} \neq 0$ in the ith column of A and let $a_{ki} \in R^*$ if R is a local ring. By Lemma 2.3 we have $\varphi(e_{i,i+1})e_{l+1,k}\varphi(e_{i,i+1}) = 0$ for $l = 1, \dots, k-2$. Hence,

$$\left(\sum_{j=1}^{n} a_{ji}e_{j,j+1}\right) e_{l+1,k} \left(\sum_{j=1}^{n} a_{ji}e_{j,j+1}\right) = a_{li}a_{ki}e_{l,k+1} \equiv 0 \bmod \mathbf{n}_{k-l+2}.$$

Thus, $a_{li}a_{ki} = 0$. In the same way, it follows from

$$e_{k-2,k-1}\varphi(e_{i,i+1})^2 = 0, \quad \varphi(e_{i,i+1})^2 e_{k+2,k+3} = 0$$

and

$$\varphi(e_{i,i+1})e_{k+1,l}\varphi(e_{i,i+1}) = 0 \quad \text{for } l = k+2, \dots, n$$

that

$$a_{k-1,i}a_{ki} = 0, \quad a_{ki}a_{k+1,i} = 0 \quad \text{and} \quad a_{ki}a_{li} = 0,$$

respectively. Hence we have

$$a_{ki}a_{li} = 0 \quad \text{for } l = 1, \dots, k-1, k+1, \dots n.$$

So by the conditions for the ring R, we have $a_{li} = 0$ for $l \neq k$. Therefore, A is monomial.

Next assume $\varphi(e_{12}) \equiv a_{k1}e_{k,k+1} \bmod \mathbf{n}_2$. By Lemma 2.1(i) we have $\varphi(e_{2,n+1}) \equiv ae_{1n} + be_{2,n+1} \bmod \mathbf{n}_n$. Then

$$\varphi(e_{1,n+1}) = \varphi([e_{12}, e_{2,n+1}]) = [a_{k1}e_{k,k+1}, ae_{1n} + be_{2,n+1}].$$

It follows from the above equality and $\varphi(e_{1,n+1}) \neq 0$ that $k = 1$ or n. □

Lemma 4.3. *There exist a graph automorphism ω and a diagonal automorphism η such that*

$$\eta^{-1}\omega^{-1}\varphi(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_2, \quad i = 1, \dots, n. \tag{4.1}$$

Proof. In view of Lemma 4.2, the matrix A is a monomial matrix, in which the sole nonzero element of the first column is a_{11} or a_{n1}. Let $\omega = \omega_0$ if $a_{n1} \neq 0$ and $\omega = 1$ if $a_{11} \neq 0$. Applying $\omega^{-1}\varphi$ to $e_{i,i+1},\ i = 1, \dots, n$, we obtain a monomial matrix, whose (1,1)-entry is not zero. For convenience sake, we still denote this matrix by $A = (a_{ji})$. We assert that A is diagonal. First show $a_{22} \neq 0$. Assume $\omega^{-1}\varphi(e_{23}) \equiv a_{k2}e_{k,k+1} \bmod \mathbf{n}_2$. Applying $\omega^{-1}\varphi$ to $[e_{12}, e_{23}] = e_{13}$ by Lemma 2.1(i) we obtain $[a_{11}e_{12}, a_{k2}e_{k,k+1}] \in \mathbf{n}_2 \backslash \mathbf{n}_3$. This implies $k = 2$ and $a_{22} \neq 0$. Repeating the argument, we obtain in turn that $a_{33}, \ldots, a_{nn}$ are all nonzero. Therefore,

$$\omega^{-1}\varphi(e_{i,i+1}) \equiv a_{ii}e_{i,i+1} \bmod \mathbf{n}_2, \quad i = 1, \ldots, n.$$

Furthermore, each $a_{ii} \in R^*$ since $\det A \in R^*$.

Set $d = \operatorname{diag}\{1, a_{11}^{-1}, (a_{11}a_{22})^{-1}, \ldots, (a_{11} \ldots a_{nn})^{-1}\}$ and $\eta = \eta_d$. Then we obtain (4.1). □

Lemma 4.4. *Let ω and η be as above. There exists an inner automorphisms σ' of $\mathbf{n}$ such that*

$$\sigma'^{-1}\eta^{-1}\omega^{-1}\varphi(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_{n-1}, \quad i = 1, \ldots, n.$$

Proof. Following the argument of Lemma 4.3, we will use induction on t to prove that there exist inner automorphisms σ_t such that

$$\begin{aligned}&\sigma_t^{-1}\eta^{-1}\omega^{-1}\varphi(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_{t+1},\\ &\quad i = 1, \ldots n, \quad t = 1, \ldots, n-2.\end{aligned} \tag{4.2}$$

Thus this shows that our lemma holds provided that we take $\sigma' = \sigma_{n-2}$.

Let $\sigma_1 = 1$. Then Lemma 4.3 shows that (4.2) is true for $t = 1$. Assume that there exists an inner automorphisms σ_{t-1}, where $1 \leqslant t-1 \leqslant n-3$, such that (4.2) is true for $t-1$. Set $\theta = \sigma_{t-1}^{-1}\eta^{-1}\omega^{-1}\varphi$ and

$$\theta(e_{i,i+1}) \equiv e_{i,i+1} + \sum_{j=1}^{n+1-t} b_{ji}e_{j,j+t} \bmod \mathbf{n}_{t+1}, \quad i = 1, \ldots, n. \tag{4.3}$$

First, using a case by case discussion, we show that on the right-hand side of (4.3),

$$b_{lk} = 0 \quad \text{for } l \neq k, k+1-t. \tag{4.4}$$

(A) $2 \leqslant l \leqslant n-t$.

(A-1) $l \leqslant n-t$ and $l \neq k-t, k-t-1$. Applying θ to $[e_{k,k+1}, e_{l+t,l+t+1}] = 0$, we have

$$\begin{aligned}&\left[e_{k,k+1} + \sum_{j=1}^{n+1-t} b_{jk}e_{j,j+t}, e_{l+t,l+t+1} + \sum_{j=1}^{n+1-t} b_{j,l+t}e_{j,j+t}\right]\\ &\quad \equiv b_{k+1,l+t}e_{k,k+1+t} - b_{k-t,l+t}e_{k-t,k+1} + b_{lk}e_{l,l+t+1} - b_{l+t+1,k}e_{l+t,l+2t+1}\\ &\quad \equiv 0 \bmod \mathbf{n}_{t+2}.\end{aligned}$$

Hence $b_{lk} = 0$.

(A-2) $l \geqslant 2$ and $l \neq k+1, k+2$. In the same way as above, applying θ to $[e_{l-1,l}, e_{k,k+1}] = 0$, we have

$$b_{lk}e_{l-1,l+t} - b_{l-1-t,k}e_{l-1-t,l} + b_{k-t,l-1}e_{k-t,k+1} - b_{k+1,l-1}e_{k,k+1+t} \equiv 0 \bmod \mathbf{n}_{t+2}.$$

Hence $b_{lk} = 0$.

(B) $l = 1$.

(B-1) $l = 1$ and $l \neq k-t, k-t-1$. This is case (A-1).

(B-2) $l = 1$ and $l = k-t$. Applying θ to $[e_{k,k+1}, [e_{k+1,k+2}, e_{k,k+1}]] = 0$, we have

$$2b_{k+2,k}e_{k,k+2+t} - b_{k-t,k}e_{k-t,k+2} \equiv 0 \bmod \mathbf{n}_{t+3}.$$

Hence $b_{k-t,k} = 0$, i.e., $b_{lk} = 0$.

(B-3) $l = 1$ and $l = k-t-1$. Applying θ to $[e_{k,k+1}, [e_{k-1,k}, e_{k,k+1}]] = 0$, we have

$$2b_{k-t-1,k}e_{k-t-1,k+1} - b_{k+1,k}e_{k-1,k+1+t} \equiv 0 \bmod \mathbf{n}_{t+3}.$$

Hence $b_{k-t-1,k} = 0$, i.e., $b_{lk} = 0$.

(C) $l = n+1-t$.

(C-1) $l = n+1-t$ and $l \neq k+1, k+2$. This is case (A-2).

(C-2) $l = n+1-t$ and $l = k+1$. As in case (B-3), we have $b_{k+1,k} = 0$, i.e., $b_{lk} = 0$.

(C-3) $l = n+1-t$ and $l = k+2$. As in case (B-2), we have $b_{k+2,k} = 0$, i.e., $b_{lk} = 0$.

Thus, (4.4) is proved and (4.3) may be rewritten as

$$\theta(e_{i,i+1}) \equiv e_{i,i+1} + b_{i+1-t,i}e_{i+1-t,i+1} + b_{ii}e_{i,i+t} \bmod \mathbf{n}_{t+1}, \quad i = 1, \ldots, n. \tag{4.5}$$

In order to complete the induction on t, we need again use induction on s to prove that there exist automorphisms $\sigma'_s,\ s = 0, 1, \ldots, n$, such that

$$\sigma_s'^{-1}\theta(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_{t+1}, \quad i = 1, \ldots, s, \tag{4.6}$$

and

$$\sigma_s'^{-1}\theta(e_{i,i+1}) \equiv e_{i,i+1} + b^{(s)}_{i+1-t,i}e_{i+1-t,i+1} + b^{(s)}_{ii}e_{i,i+t} \bmod \mathbf{n}_{t+1}, \quad i = s+1, \ldots, n. \tag{4.7}$$

Let $\sigma'_0 = 1$. Then (4.7) with $b^{(0)}_{i+1-t,i} = b_{i+1-t,i}$ and $b^{(0)}_{ii} = b_{ii}$ is trivially true by (4.5), and (4.6) does not occur. Assume that (4.6) and (4.7) hold for some inner automorphism σ'_{s-1} with $1 \leqslant s \leqslant n$. In particular,

$$\sigma_{s-1}'^{-1}\theta(e_{s,s+1}) \equiv e_{s,s+1} + b^{(s-1)}_{s+1-t,s}e_{s+1-t,s+1} + b^{(s-1)}_{ss}e_{s,s+t} \bmod \mathbf{n}_{t+1}. \tag{4.8}$$

In fact, $b^{(s-1)}_{s+1-t,s} = 0$. in (4.8). For applying $\sigma_{s-1}'^{-1}\theta$ to $[e_{s-t,s-t+1}, e_{s,s+1}] = 0$, we have

$$b^{(s-1)}_{s+1-t,s}e_{s-t,s+1} \equiv 0 \bmod \mathbf{n}_{t+2}.$$

Hence, $b^{(s-1)}_{s+1-t,s} = 0$. Set $z = e - b^{(s-1)}_{ss}e_{s+1,s+t}$ and $\sigma'_s = \sigma'_{s-1}\sigma_z$. Then for $1 \leqslant i \leqslant s$ we have

$$\sigma'^{-1}_s\theta(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_{t+1}$$

and for $i > s$,

$$\sigma'^{-1}_s\theta(e_{i,i+1}) \equiv e_{i,i+1} + b^{(s)}_{i+1-t,i}e_{i+1-t,i+1} + b^{(s)}_{ii}e_{i,i+t} \bmod \mathbf{n}_{t+1},$$

where $b^{(s)}_{i+1-t,i} = b^{(s-1)}_{i+1-t,i} + \delta_{s+t,i}b^{(s-1)}_{ss}$ and $b^{(s)}_{ii} = b^{(s-1)}_{ii}$.

Thus, there exists an inner automorphism σ'_n such that (4.6) holds. Hence, (4.2) is true for t if we take $\sigma_t = \sigma_{t-1}\sigma'_n$. The proof is completed. □

Lemma 4.5. *Let ω, η and σ' be as above. There exist an inner automorphism σ'', an extremal automorphism ξ and a central automorphisms μ of $\mathbf{n}$ such that*

$$\mu^{-1}\xi^{-1}\sigma''^{-1}\sigma'^{-1}\eta^{-1}\omega^{-1}\varphi(e_{i,i+1}) = e_{i,i+1}, \quad i = 1, \ldots, n.$$

Proof. Set $\theta = \sigma'^{-1}\eta^{-1}\omega^{-1}\varphi$. By Lemma 4.4 we have

$$\theta(e_{i,i+1}) \equiv e_{i,i+1} + c_{1i}e_{1n} + c_{2i}e_{2,n+1} \bmod \mathbf{n}_n, \quad i = 1, \ldots, n.$$

For $2 < i < n$ applying θ to $[e_{12}, e_{i,i+1}] = 0$, we have $c_{2i}e_{1,n+1} = 0$, from which it follows that $c_{2i} = 0$. Similarly for $1 < i < n-1$ applying θ to $[e_{i,i+1}, e_{n,n+1}] = 0$, we have $c_{1i} = 0$. Furthermore, $\theta([e_{12}, e_{n,n+1}]) = 0$ implies that $c_{2n} = -c_{11}$. Thus, we have

$$\theta(e_{12}) \equiv e_{12} + c_{11}e_{1n} + c_{21}e_{2,n+1} \bmod \mathbf{n}_n,$$

$$\theta(e_{n,n+1}) \equiv e_{n,n+1} + c_{1n}e_{1n} - c_{11}e_{2,n+1} \bmod \mathbf{n}_n,$$

$$\left.\begin{array}{l} \theta(e_{23}) \equiv e_{23} + c_{22}e_{2,n+1} \bmod \mathbf{n}_n \\ \theta(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_n, \quad i = 3, \ldots, n-2 \\ \theta(e_{n-1,n}) \equiv e_{n-1,n} + c_{1,n-1}e_{1n} \bmod \mathbf{n}_n \end{array}\right\} \quad \text{if } n > 3,$$

and

$$\theta(e_{23}) \equiv e_{23} + c_{12}e_{13} + c_{22}e_{24} \bmod \mathbf{n}_3 \quad \text{if } n = 3.$$

Set $z = e + c_{1,n-1}e_{1,n-1} - c_{11}e_{2n} - c_{22}e_{3,n+1}$. Then

$$\sigma_z^{-1}\theta(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_n, \quad i = 2, \ldots, n-1,$$

$$\sigma_z^{-1}\theta(e_{12}) \equiv e_{12} + c_{21}e_{2,n+1} \bmod \mathbf{n}_n$$

and

$$\sigma_z^{-1}\theta(e_{n,n+1}) \equiv e_{n,n+1} + c_{1n}e_{1n} \bmod \mathbf{n}_n.$$

Set $b = (c_{21}, c_{1n})$ and $\xi = \xi_b$. Then $\xi^{-1}\sigma_z^{-1}\theta$ acts trivially on $e_{i,i+1} \bmod \mathbf{n}_n$ for $i = 1, \ldots, n$. Assume that

$$\xi^{-1}\sigma_z^{-1}\theta(e_{i,i+1}) = e_{i,i+1} + c_i e_{1,n+1}, \quad i = 1, \ldots, n.$$

Set $z' = e + c_n e_{1n} - c_1 e_{2,n+1}$. Then $\sigma_{z'}^{-1}\xi^{-1}\sigma_z^{-1} = \xi^{-1}\sigma''^{-1}$ with some $\sigma'' \in \mathscr{I}$ and

$$\xi^{-1}\sigma''^{-1}\theta(e_{i,i+1}) = e_{i,i+1} + c_i e_{1,n+1}, \quad i = 2, \ldots, n-1,$$

$$\xi^{-1}\sigma''^{-1}\theta(e_{12}) = e_{12} \quad \text{and} \quad \xi^{-1}\sigma''^{-1}\theta(e_{n,n+1}) = e_{n,n+1}.$$

Let $\mu = \mu_c$ be the central automorphism of $\mathbf{n}$ associated to $c = (c_2, \ldots, c_{n-1})$. Then $\mu^{-1}\xi^{-1}\sigma''^{-1}\theta$ acts trivially on $e_{i,i+1}$ for $i = 1, \ldots, n$. Thus the proof is completed. $\square$

Proof of Theorem 4.1. By Lemma 4.5 we have

$$\mu^{-1}\xi^{-1}\sigma'''^{-1}\eta^{-1}\omega^{-1}\varphi(e_{i,i+1}) = e_{i,i+1}, \quad i = 1, \ldots, n,$$

where $\sigma''' = \sigma'\sigma''$. Since $e_{i,i+1},\ i = 1, \ldots, n$, generate the Lie algebra $\mathbf{n}$, we have

$$\mu^{-1}\xi^{-1}\sigma'''^{-1}\eta^{-1}\omega^{-1}\varphi = 1$$

and so $\varphi = \omega \cdot \eta \cdot \sigma''' \cdot \xi \cdot \mu = \omega \cdot \eta \cdot \xi \cdot \mu \cdot \sigma$ for some $\sigma \in \mathscr{I}$. $\square$

The following theorem gives a more precise description of the automorphism group of $\mathbf{n}$, which shows that the five types of standard automorphisms of $\mathbf{n}$ are all necessary for building the automorphism group of $\mathbf{n}$ and for any automorphism φ of $\mathbf{n}$, the decomposition $\varphi = \omega \cdot \eta \cdot \xi \cdot \mu \cdot \sigma$ in Theorem 4.1 is unique.

Theorem 4.6. *Assume that $n \geqslant 3$ and R is as in Theorem* 4.1. *Then* $\mathrm{Aut}(\mathbf{n}) = \mathscr{G} \ltimes (\mathscr{D} \ltimes ((\mathscr{E} \times \mathscr{C}) \ltimes \mathscr{I}))$.

Proof. It is clear that the product of the extremal automorphism group $\mathscr{E}$ and the central automorphism group $\mathscr{C}$ is a direct product. Since $\mathscr{I} \lhd \mathrm{Aut}(\mathbf{n})$, the product $(\mathscr{C} \times \mathscr{E})\mathscr{I}$ is a group. Assume that $\sigma_x = \xi_b\mu_c$ is in $(\mathscr{C} \times \mathscr{E}) \cap \mathscr{I}$, where $x = e + \sum_{i<j} a_{ij}e_{ij},\ \ b = (b_1, b_2)$ and $c = (c_2, \ldots, c_{n-1})$. Then we have $xe_{k,k+1}x^{-1} = \xi_b\mu_c(e_{k,k+1})$, i.e., $xe_{k,k+1} = (\xi_b\mu_c(e_{k,k+1}))x$ for $k = 1, \ldots, n$. Hence for $2 \leqslant k \leqslant n-1$ we have

$$\sum_{i<k} a_{ik}e_{i,k+1} = \sum_{j>k+1} a_{k+1,j}e_{kj} + c_k e_{1,n+1},$$

which implies that $c_k = 0$. Similarly, we have $b_l = 0$ for $l = 1, 2$. Hence $\xi_b\mu_c = 1$ and so $(\mathscr{E} \times \mathscr{C})\mathscr{I} = (\mathscr{E} \times \mathscr{C}) \ltimes \mathscr{I}$. It is not difficult to see that the diagonal automorphism group $\mathscr{D}$ normalizes $\mathscr{E}, \mathscr{C}$ and $\mathscr{I}$. So $\mathscr{D}((\mathscr{E} \times \mathscr{C}) \ltimes \mathscr{I})$ is a subgroup of $\mathrm{Aut}(\mathbf{n})$. Obviously any automorphism in $(\mathscr{E} \times \mathscr{C}) \ltimes \mathscr{I}$ acts trivially on $e_{i,i+1} \bmod \mathbf{n}_2$, $i = 1, \ldots, n$. Hence, $\mathscr{D} \cap ((\mathscr{E} \times \mathscr{C}) \ltimes \mathscr{I}) = 1$, which implies that $\mathscr{D}((\mathscr{E} \times \mathscr{C}) \ltimes \mathscr{I})$

$= \mathscr{D}\ltimes((\mathscr{E}\times\mathscr{C})\ltimes\mathscr{I})$. Finally, since the graph automorphism ω_0 normalizes each of the subgroups $\mathscr{D}$, $\mathscr{E}$, $\mathscr{C}$ and $\mathscr{I}$, $\mathscr{G}(\mathscr{D}\ltimes((\mathscr{E}\times\mathscr{C})\ltimes\mathscr{I}))$ is a subgroup of Aut($\mathbf{n}$). Moreover, $\omega_0(e_{12}) \notin Re_{12} \bmod \mathbf{n}_2$. But for any φ in $\mathscr{D}\ltimes((\mathscr{E}\times\mathscr{C})\ltimes\mathscr{I})$, $\varphi(e_{12}) \in Re_{12} \bmod \mathbf{n}_2$. So the intersection $\mathscr{G}\cap(\mathscr{D}\ltimes((\mathscr{E}\times\mathscr{C})\ltimes\mathscr{I})) = 1$. Therefore, $\mathscr{G}(\mathscr{D}\ltimes((\mathscr{E}\times\mathscr{C})\ltimes\mathscr{I})) = \mathscr{G}\ltimes(\mathscr{D}\ltimes((\mathscr{E}\times\mathscr{C})\ltimes\mathscr{I}))$. By Theorem 4.1 we complete the proof. □

5. The automorphism group of n for $n = 1, 2$

Theorem 5.1. *When $n = 1$, any automorphism of* $\mathbf{n}$ *is diagonal and* Aut($\mathbf{n}$) $\cong R^*$.

Proof. Let $\varphi \in$ Aut($\mathbf{n}$) and $\varphi(e_{12}) = ae_{12}$. By Lemma 2.1(ii) we have $a \in R^*$. It is clear that $\varphi = \eta_x$, where $x = \operatorname{diag}\{a, 1\}$. Hence the first assertion is true. By Lemma 3.3 we have Aut($\mathbf{n}$) = $\mathscr{D} \cong D/R^*e \cong R^*$. □

When $n = 2$, we will only discuss the case that R is a local commutative ring; however the restriction that 2 is a unit will be omitted. In this case all central automorphisms are improper.

Theorem 5.2. *Let $n = 2$ and R be any local commutative ring. Let φ be an arbitrary automorphism of* $\mathbf{n}$. *Then there are graph, diagonal, extremal and inner automorphisms ω, η, ξ and σ, respectively, of* $\mathbf{n}$ *such that $\varphi = \omega\cdot\eta\cdot\xi\cdot\sigma$.*

Proof. Let A be the 2×2 matrix as in Lemma 2.2. Then $a_{11}a_{22} - a_{12}a_{21} \in R^*$. Since R is a local ring, a_{11} and a_{22} are both in R^* or a_{12} and a_{21} are both in R^*. Let $\omega = 1$ if $a_{11}, a_{22} \in R^*$ and $\omega = \omega_0$ if $a_{12}, a_{21} \in R^*$. Applying $\omega^{-1}\varphi$ to e_{12} and e_{23}, we obtain a 2×2 matrix as in Lemma 2.2, which is still denoted by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

For this matrix we have $a_{11}, a_{22} \in R^*$. Set $\eta = \eta_d$ with $d = \{a_{11}, 1, a_{22}^{-1}\}$. Then we have

$$\eta^{-1}\omega^{-1}\varphi(e_{12}) \equiv e_{12} + a_{22}^{-1}a_{21}e_{23} \bmod \mathbf{n}_2$$

and

$$\eta^{-1}\omega^{-1}\varphi(e_{23}) \equiv a_{11}^{-1}a_{12}e_{12} + e_{23} \bmod \mathbf{n}_2.$$

Set $\xi = \xi_b$ with $b = (b_1, b_2) = (a_{22}^{-1}a_{21}, a_{11}^{-1}a_{12})$. Since $a_{11}a_{22} - a_{12}a_{21} \in R^*$, we have $d = 1 - b_1b_2 \in R^*$ and so ξ is indeed an extremal automorphism of $\mathbf{n}$. It is not difficult to check that

$$\xi^{-1}(e_{12}) \equiv \frac{1}{d}e_{12} - \frac{b_1}{d}e_{23} \bmod \mathbf{n}_2$$

and

$$\xi^{-1}(e_{23}) \equiv -\frac{b_2}{d}e_{12} + \frac{1}{d}e_{23} \bmod \mathbf{n}_2.$$

Therefore,

$$\xi^{-1}\eta^{-1}\omega^{-1}\varphi(e_{i,i+1}) \equiv e_{i,i+1} \bmod \mathbf{n}_2, \quad i = 1, 2.$$

Assume

$$\xi^{-1}\eta^{-1}\omega^{-1}\varphi(e_{i,i+1}) = e_{i,i+1} + a_i e_{13}, \quad i = 1, 2.$$

Set $\sigma = \sigma_x$ with $x = e + a_2 e_{12} - a_1 e_{23}$. Then $\sigma^{-1}\xi^{-1}\eta^{-1}\omega^{-1}\varphi$ acts trivially on $e_{12},\ e_{23}$, and so it is the identity automorphism of $\mathbf{n}$. Thus $\varphi = \omega \cdot \eta \cdot \xi \cdot \sigma$. $\square$

Remarks. (i) The decomposition $\varphi = \omega \cdot \eta \cdot \xi \cdot \sigma$ in Theorem 5.2 is not unique. For example, let R be a field of characteristic other than 2. Let $b = (2, 1),\ b' = (1, \frac{1}{2})$ and $d = \mathrm{diag}\{-2, 1, -1\}$. It is trivial to check $\omega_0 \cdot \xi_b = \eta_d \cdot \xi_{b'}$.

(ii) If R is an integral domain, then Theorem 5.2 is false. For example, let R be the ring of integers and $\varphi \in \mathrm{Aut}(\mathbf{n})$ such that $\varphi(e_{12}) = 2e_{12} + 3e_{23}$ and $\varphi(e_{23}) = 3e_{12} + 5e_{23}$. Since any product $\omega \cdot \eta \cdot \xi \cdot \sigma$ maps e_{12} to $ae_{12} + be_{23} \bmod \mathbf{n}_2$ with $a \in R^*$ or $b \in R^*$, 2 and 3 are both not in R^*. Thus, Theorem 5.2 is not true for φ.

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